THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2050A Mathematical Analysis I (Fall 2021)
Suggested Solution of Test 1

If you find any errors or typos, please email me at
yzwang@math.cuhk.edu.hk

1. (21 points) (a) State the Archimedean property;

   (b) State the definition of $\sup S$ for a non-empty subset of $\mathbb{R}$;

   (c) State the completeness of $\mathbb{R}$;

Solution:

(a) The set of natural number $\mathbb{N}$ is unbounded.

(b) Given a non-empty subset $S \subset \mathbb{R}$. A real number $u = \sup S$ if $s \leq u$ for any $s \in S$ and $s \leq v$ for any $s \in S$ implies $v \geq u$.

(c) For any non-empty subset $S \subset \mathbb{R}$ which is bounded from above, $\sup S$ exists.

2. (25 points) Let A and B be two bounded subset of $\mathbb{R}$. Define $A+B=\{a+b : a \in A, b \in B\}$.

   (a) Show that both $\sup(A+B)$ and $\inf(A+B)$ exist;

   (b) Show that

$$\sup(A+B) = \sup A + \sup B, \quad \text{and} \quad \inf(A+B) = \inf A + \inf B.$$

Solution:

(a) Since A, B are bounded, there exist $\sup A, \sup B, \inf A, \inf B \in \mathbb{R}$ such that for any $a \in A$ and $b \in B$,

$$\inf A \leq a \leq \sup A, \quad \inf B \leq b \leq \sup B.$$

It follows that for any $x \in A+B$ where $x = a+b$ for some $a \in A$ and $b \in B$,

$$\inf A + \inf B \leq x \leq \sup A + \sup B.$$

Therefore $A+B$ is bounded. By completeness of $\mathbb{R}$, we have that $\sup(A+B)$ and $\inf(A+B)$ exist.

(b) From (a), we have that $A+B$ is bounded above by $\sup A + \sup B$.

Suppose $A+B$ is bounded above by some $u \in \mathbb{R}$. Then for any $a \in A$ and $b \in B$,

$$a+b \leq u.$$

Now fix $b_0 \in B$, we have that $a \leq u - b_0$ for any $a \in A$. Then $u - b_0$ is an upper bound of A and

$$\sup A \leq u - b_0.$$

Since b_0 is arbitrarily chosen, we can conclude that $b \leq u - \sup A$ for any $b \in B$. Then $u - \sup A$ is an upper bound of B and

$$\sup B \leq u - \sup A.$$

In sum, $A+B$ is bounded above by $\sup A + \sup B$ and $\sup A + \sup B \leq u$ for any upper bound u of $A+B$. Therefore, $\sup A + \sup B = \sup(A+B)$.

From (a), we have that $A+B$ is bounded blow by $\inf A + \inf B$.

Suppose $A+B$ is bounded blow by some $v \in \mathbb{R}$. Then for any $a \in A$ and $b \in B$,

$$a+b \geq v.$$

Now fix $b_0 \in B$, we have that $a \geq v - b_0$ for any $a \in A$. Then $v - b_0$ is an lower bound of A and

$$\inf A \geq v - b_0.$$

Since b_0 is arbitrarily chosen, we can conclude that $b \geq v - \inf A$ for any $b \in B$. Then $v - \inf A$ is an lower bound of B and

$$\inf B \geq v - \inf A.$$

In sum, $A + B$ is bounded below by $\inf A + \inf B$ and $\inf A + \inf B \geq v$ for any lower bound v of $A + B$. Therefore, $\inf A + \inf B = \inf(A + B)$.

3. (29 points) (a) Give the definition of $\lim_{n\to+\infty} a_n = a$;

(b) State the negation of $\lim_{n\to+\infty} a_n = a$;

(c) Determine the convergence of the sequence $\{\frac{1}{n} - \frac{1}{n+1}\}_{n=1}^{\infty}$. Justify your answer.

Solution:

(a) For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|a_n - a| < \epsilon$ whenever $n > N$.

(b) There exists $\epsilon_0 > 0$ such that for any $N \in \mathbb{N}$, we can find $n > N$ such that $|a_n - a| \geq \epsilon_0$.

(c) Let $a_n = \frac{1}{n} - \frac{1}{n+1}$. For any $\epsilon > 0$, we choose $N \in \mathbb{N}$ such that $N \geq \frac{1}{\sqrt{\epsilon}}$. Then for $n > N$, we have that

$$|a_n - 0| = \frac{1}{n(n+1)} < \frac{1}{N(N+1)} < \frac{1}{N^2} \leq \epsilon.$$

Hence $\lim_{n\to+\infty} a_n = 0$.

4. (25 points) Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence of real number such that $\lim_{n\to+\infty} a_n = a$.

   (a) If $a_n \in (0,1)$ for all $n \in \mathbb{N}$, show that $a \in [0,1]$.

   (b) Suppose that $a > 0$, is it true that $\lim_{n\to+\infty} a_n^{\frac{1}{n}} = 1$? If yes, prove it. Otherwise, provides a counterexample.

Solution:

(a) Suppose $a < 0$ or $a > 1$. Then there exists an $\epsilon_0 > 0$ such that $a \leq -\epsilon_0$ or $a \geq 1 + \epsilon_0$. Since $\lim_{n\to+\infty} a_n = a$, for this ϵ_0, we can find $N_0 \in \mathbb{N}$ such that $|a_{N_0} - a| < \epsilon_0$. Then $a_{N_0} \leq 0$ or $a_{N_0} \geq 1$, contradicting the condition that $a_n \in (0,1)$ for all $n \in \mathbb{N}$. Therefore, $0 \leq a \leq 1$.

(b) Fix $\epsilon_0 > \max(0, a-1)$.

Since $\lim_{n\to+\infty} a_n = a$, there exists $N_0 \in \mathbb{N}$ such that

$$a - \epsilon_0 < a_n < a + \epsilon_0$$

for $n > N_0$ and $a - \epsilon_0 < 1$.

Let $b_n = a_n^{\frac{1}{n}}$. Then for $n > N_0$,

$$(a-\epsilon_0)^{\frac{1}{n}} < b_n < (a+\epsilon_0)^{\frac{1}{n}}.$$

Since $a_n - 1 = b_n^n - 1 = (b_n - 1)(b_n^{n-1} + b_n^{n-2} + \cdots + b_n + 1)$, we have that for $n > N_0$,

$$|b_n - 1| = \frac{|a_n - 1|}{|b_n^{n-1} + b_n^{n-2} + \cdots + b_n + 1|} < \frac{|a_n| + 1}{n(a-\epsilon_0)^{\frac{n-1}{n}}} < \frac{a + \epsilon_0 + 1}{a - \epsilon_0} \cdot \frac{1}{n}.$$

By Theorem 1.1 in quick note of Week 3, since $\lim_{n\to+\infty} \frac{1}{n} = 0$, we have that

$$\lim_{n\to+\infty} a_n^{\frac{1}{n}} = \lim_{n\to+\infty} b_n = 1.$$